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On the Motion of Space Charge in a Dielectric Medium

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ON THE MOTION OF SPACE CHARGE IN A DIELECTRIC MEDIUM

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CONTENTS

A general relation is derived which describes the spatio-temporal behaviour of space charge in an ideal dielectric medium. (Diffusion effects are neglected.) This result is used to determine the behaviour of a parallel plate capacitor which contains space charges near its electrodes. Expressions are derived for the variation with time of the potential difference between the plates following the removal of an applied voltage and for the external current which flows when the plates are connected together. Symmetrical and asymmetrical charge distributions are considered.

1. Introduction

Although an ideal dielectric has no intrinsic electrical conductivity, a conduction current does in fact flow when a steady potential difference is applied to a real dielectric specimen. It is very usual for this current to decay to a steady value from a higher initial value. If the applied voltage is suddenly removed, electrical effects are observable for some time afterwards. For example, if the specimen is left on open circuit, a residual potential difference is detectable which decays with time; if the specimen is short circuited, a current is observable which flows in the opposite direction to the earlier steady current and which decays with time. There are several variations on these observations. For example, it has been observed (Yahagi, Kao & Calderwood 1966) that if a specimen of *n*-hexane is irradiated, the current builds up rather than decays to its steady value, while on switching off, the current which flows into a short circuit is in the same direction as the original current; however, a short pulse of current in the opposite direction is sometimes detected immediately the short circuit connexion is made.

Various explanations (Adamczewski 1969) have been put forward to account for this type of behaviour, and many depend on the postulate that space charge exists in the dielectric medium. This space charge is usually considered to be concentrated near the electrodes, while a steady current is flowing; some time is required after the application of a voltage for an equilibrium state to be reached, and the space charge is imagined to move in such a way after the applied voltage is removed as to cause the subsequent observations of current and voltage.

Although these explanations are dependent upon space charge and its movement, usually only a qualitative consideration is given to the action supposed to be taking place. The aim of the present work is to consider the movement of space charges, and the effects to which they give rise, in a more accurate and quantitative way.

To this end we study in the following sections a particularly simple model consisting of two parallel conducting plates immersed in a dielectric medium and having space charge distributed between them. We first derive, in § 2, a general equation for the spatial and temporal dependence of electric charge density in a non-conducting fluid. For ease of mathematical manipulation diffusion and dielectric relaxation effects are ignored. In § 3 we deal with the case of a symmetrical space charge distribution while in § 4 we consider a particular unsymmetrical distribution. The results obtained in §§ 3 and 4 are discussed in § 5.

2. General considerations

Consider positive and negative charges, distributed with number densities $n^+(\boldsymbol{r}, t)$ and $n^-(\boldsymbol{r}, t)$, in an insulating medium of absolute permittivity, ϵ†. The time variations of these densities are governed by the continuity equations

$$\partial n^+(\boldsymbol{r}, t)/\partial t = G(\boldsymbol{r}, t) - R(\boldsymbol{r}, t) - \operatorname{div} \boldsymbol{j}^+(\boldsymbol{r}, t), \tag{2.1a}$$

$$\partial n^-(\boldsymbol{r}, t)/\partial t = G(\boldsymbol{r}, t) - R(\boldsymbol{r}, t) - \operatorname{div} \boldsymbol{j}^-(\boldsymbol{r}, t). \tag{2.1b}$$

The rates, per unit volume, at which charge pairs are generated and recombine are denoted by $G(\boldsymbol{r}, t)$ and $R(\boldsymbol{r}, t)$ respectively. The *particle* current densities, $\boldsymbol{j}^+(\boldsymbol{r}, t)$ and $\boldsymbol{j}^-(\boldsymbol{r}, t)$ are related to the particle velocities, $\boldsymbol{v}^+(\boldsymbol{r}, t)$ and $\boldsymbol{v}^-(\boldsymbol{r}, t)$, by the equations

$$\boldsymbol{j}^+ = n^+\boldsymbol{v}^+ \quad \text{and} \quad \boldsymbol{j}^- = n^-\boldsymbol{v}^-. \tag{2.2}$$

If the magnitude of the charge on each particle is q, then the *net* charge density, $\rho = q(n^+ - n^-)$, is related to the total electric current density $\boldsymbol{J} = q(\boldsymbol{j}^+ - \boldsymbol{j}^-)$, by the formula

$$\partial \rho/\partial t = -\operatorname{div} \boldsymbol{J}, \tag{2.3}$$

obtained by subtracting equation (2.1 *b*) from equation (2.1 *a*).

In general the charged particles will be caused to move by two influences, namely, the electric potential gradient, $-\boldsymbol{E}$ (which is determined not just by ρ only, but also by whatever external charge distributions there may be) and the density gradients $\operatorname{grad} n^+$ and $\operatorname{grad} n^-$. In fact

$$\boldsymbol{j}^+ = \mu^+ n^+ \boldsymbol{E} - d^+ \operatorname{grad} n^+$$

and

$$\boldsymbol{j}^- = -\mu^- n^- \boldsymbol{E} - d^- \operatorname{grad} n^-,$$

where $\mu^\pm$ denotes the mobility and $d^\pm$ the diffusion coefficient. Throughout this paper we shall neglect diffusion effects. Consequently,

$$\boldsymbol{J} = q(\mu^+ n^+ + \mu^- n^-)\,\boldsymbol{E}, \tag{2.4}$$

and this result with equation (2.3) yields the following expression for the time rate of change of the charge density at a particular point $\boldsymbol{r}$:

$$\partial \rho(\boldsymbol{r}, t)/\partial t = -q \operatorname{div}\{[\mu^+ n^+ + \mu^- n^-]\,\boldsymbol{E}\}. \tag{2.5}$$

† SI units are used in this paper.

The net charge density, ρ, and the potential gradient, $-\boldsymbol{E}$, are not independent because of the relation

$$\operatorname{div}\boldsymbol{D} = \operatorname{div}\epsilon\boldsymbol{E} = \rho,$$

and so equation (2.5) may be written

$$\partial\rho/\partial t = -q(\mu^+n^+ + \mu^-n^-)(\rho/\epsilon) - q(\boldsymbol{E}.\nabla)(\mu^+n^+ + \mu^-n^-). \tag{2.6}$$

For later calculations it is useful to know the time variation of ρ as the point of observation moves with a charge of a particular sign. If the operator $\mathscr{D}^+$, defined by the equation

$$\mathscr{D}^+ = \partial/\partial t + (\boldsymbol{v}^+.\nabla),$$

is applied to ρ we obtain the variation, with time, of the charge density in the vicinity of a moving positive charge. Thus

$$\mathscr{D}^+\rho = \partial\rho/\partial t + (\boldsymbol{v}^+.\nabla)(qn^+ - qn^-),$$

and with equation (2.6) we obtain

$$\mathscr{D}^+\rho = -(\mu^+\rho^2/\epsilon) - q(\mu^+ + \mu^-)\nabla.(n^-\boldsymbol{E}). \tag{2.7}$$

In a region of space where there are charged particles of one sign only, positive say, this last equation demands that

$$\mathscr{D}^+\rho = -(\mu^+\rho^2/\epsilon).$$

Integration of this equation with respect to time leads to the result

$$\rho(\boldsymbol{r}', t') = \rho(\boldsymbol{r}, t)/[1 + \mu^+\rho(\boldsymbol{r}, t)(t'-t)/\epsilon]. \tag{2.8}$$

In this equation the period $(t'-t)$ is the time taken for a positive charge, initially at $\mathbf{r}$ at time t, to move to the position $\boldsymbol{r}'$. Equation (2.8) is of primary importance in our later considerations.

3. Symmetrical charge distribution

We shall consider here the time dependence of the spatial distribution of an electric space charge placed between two parallel conducting plates immersed in an insulating fluid of absolute permittivity, ϵ.

We shall suppose that before time $t = 0$ a charging current is flowing through the fluid from left to right, which we shall deem to be the positive direction of current flow. This current flow is maintained by the application of a potential difference between the plates; this potential difference sets up a field in the positive direction, by creating a positive surface charge density on the left-hand plate, and a negative surface charge density on the right-hand plate. We shall assume that, possibly because of difficulty in the discharge of the charge carriers in the fluid when the plates are reached, there is a build up of negative charge near the left-hand plate, and of positive charge near the right-hand plate.

At time $t = 0$, we shall suppose that there are space charges near each plate. These oppositely charged space charges extend from the inner surface of each plate to a distance l_0 into the fluid. The space charge density, ρ, has, initially, the same magnitude, ρ_0, at every point in the intervals $-L \leqslant x \leqslant -(L-l_0)$ and $(L-l_0) \leqslant x \leqslant L$, as illustrated in figure 1. The separation of the plates, $2L$, is small enough to allow the neglect of edge effects. Furthermore, it is assumed that at time zero there are also surface charge densities, $+\sigma_0$ and $-\sigma_0$ on the inner surfaces of the left- and right-hand plates, respectively. The initial values l_0, ρ_0 and σ_0 are not independent because we shall require the left-hand plate to be at some positive potential, $V_{PQ}(0)$, relative to the right-hand plate.

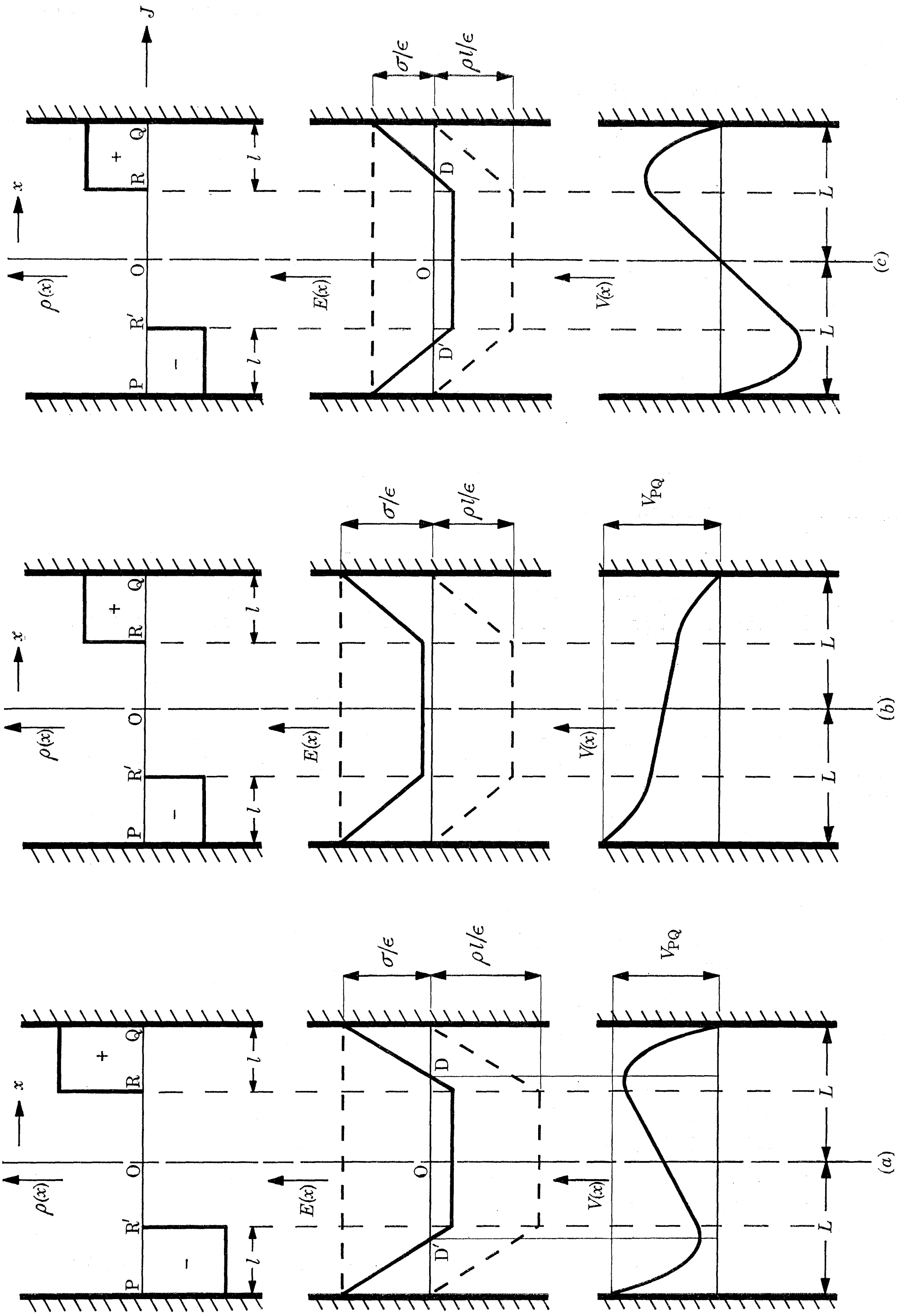

FIGURE 1. Distribution of charge, electric field and potential for symmetrical case: (*a*) open-circuit, $s_0 < 1$; (*b*) open-circuit, $s_0 > 1$; (*c*) short-circuit.

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With appropriate values of l_0, ρ_0 and σ_0, we shall study, in § 3(*a*), the temporal evolution of this system after it has been disconnected from the external potential source and, in § 3*b*, we shall deal with the case when the plates are connected together immediately after time zero.

In the first case the quantity of experimental importance is the potential difference between the plates, whereas in the second case it is the current flowing in the external connexion between the plates that is of interest.

3 (*a*). *Open-circuit case*

The initial conditions of this problem are indicated in figures 1*a* and 1*b*. In § 2 we have shown that the spatio-temporal variation of the density of charge of one sign in an insulating fluid is governed by the equation (with an appropriate change in notation for the one dimensional case)

$$\rho(x', t') = \rho(x, t)/[1 + \mu\rho(x, t)\,(t' - t)/\epsilon], \qquad (3\,(a)\,1)$$

where $(t' - t)$ is the time taken for a charged particle to move from x to x'. For simplicity we shall assume that the positive and negative charges have the same value of mobility, μ. It follows from equation (3 (*a*) 1) that because the space charges are initially uniformly distributed they remain so. Thus the space charge distributions maintain their rectangular shape and the charge density, $\rho(t)$, at time t, satisfies the equation

$$\left.\begin{aligned}\rho(t) &= \rho_0/(1+\gamma t),\\ \text{where}\qquad \rho_0 = \rho(0) \quad &\text{and} \quad \gamma = \mu\rho_0/\epsilon.\end{aligned}\right\} \qquad (3\,(a)\,2)$$

The potential difference between the left- and right-hand plates, $V_{PQ}(t)$, may be expressed in terms of l, L, ρ and σ by the equation

$$\epsilon V_{PQ}(t) = 2L\sigma - (2L - l)\,\rho l. \qquad (3\,(a)\,3)$$

In order to determine the time dependence of $V_{PQ}(t)$ we must investigate the variation with time of both σ and l.

The rate of decrease of surface charge density on a particular plate must be equal to the current density entering that plate. Therefore,

$$\mathrm{d}\sigma/\mathrm{d}t = -\mu E(L, t)\,\rho(t), \qquad (3\,(a)\,4)$$

where $E(L, t) = \sigma/\epsilon$ is the electric field at the surface of the right-hand plate. Equation (3 (*a*) 2) combined with equation (3 (*a*) 4) implies that

$$\mathrm{d}\sigma/\mathrm{d}t = -\gamma\sigma/(1+\gamma t),$$

and therefore

$$\sigma(t) = \sigma_0/(1+\gamma t). \qquad (3\,(a)\,5)$$

The rate of increase of $l(t)$ with time is determined by the velocity of the charges at the edge of the charge cloud at R, namely $\mu E(L - l, t)$. Consequently

$$\mathrm{d}l/\mathrm{d}t = \mu E(L-l, t) = (\mu/\epsilon)\,[\rho(t)\,l(t) - \sigma(t)],$$

which, with equations (3 (*a*) 2) and (3 (*a*) 5) reduces to

$$\mathrm{d}l/\mathrm{d}t = \gamma[l(t) - (\sigma_0/\rho_0)]/(1+\gamma t),$$

and hence, setting $s_0 = (\sigma_0/\rho_0 l_0)$ we have

$$l(t) = l_0[1 + (1 - s_0)\,\gamma t]. \qquad (3\,(a)\,6)$$

The range of t for which this equation is valid depends upon the value of s_0. For $s_0 > 1$, equation (3 (*a*) 6) indicates that $l(t)$ decreases with time which means that the points R′ and R diverge (figure 1*b*). When $t^{-1} = \gamma(s_0-1)$, $l(t)$ is zero and the equation is not valid for greater values of t. When $s_0 = 1$, $l(t)$ is constant; therefore, only the magnitude of the density of the space charge varies with time, whereas its spatial extension remains unchanged. From equation (3 (*a*) 3) we conclude that for $V_{PQ}(0)$ to be greater than zero $s_0 > [1-(l_0/2L)]$ and so values of s_0 less than unity are compatible with the requirement that $V_{PQ}(0) > 0$ (see figure 1*a*). In this instance $l(t)$ is an increasing function of time and the points R′ and R converge. When t attains the value $(L-l_0)/[l_0\gamma(1-s_0)]$, R′ and R coincide and thereafter equation (3 (*a*) 6) is invalid. These remarks may be summarized by saying that equation (3 (*a*) 6) holds under the following conditions:

$$\left.\begin{aligned} &0 \leqslant t \leqslant [1/\gamma(s_0-1)] = T_c, \quad s_0 > 1;\\ &0 \leqslant t \leqslant \infty, \quad s_0 = 1;\\ &0 \leqslant t \leqslant [(L-l_0)/l_0\gamma(1-s_0)] = T_e, \quad 1-(l_0/2L) < s_0 < 1.\end{aligned}\right\} \tag{3 (a) 7}$$

Combining equations (3 (*a*) 2), (3 (*a*) 3), (3 (*a*) 5) and (3 (*a*) 6) we obtain the relation

$$\epsilon V_{PQ}(t) = \frac{2L\rho_0 l_0}{1+\gamma t}\{s_0-[1+(1-s_0)\gamma t]+(l_0/2L)[1+(1-s_0)\gamma t]^2\}. \tag{3 (a) 8}$$

The range of values of t and of s_0 for which this equation is valid are those indicated in (3 (*a*) 7). If $s_0 > 1$, the charge clouds contract into the plates and this process continues until a time $T_c = [1/\gamma(s_0-1)]$ when the space charges will have been annihilated. A charge density of amount $(\sigma_0-\rho_0 l_0) = \sigma_0(s_0-1)/s_0$ will remain on the plates after this time in view of the assumed infinite intrinsic resistivity of the fluid. An immediate consequence is that after time T_c the potential V_{PQ} remains constant. If we set $(s_0-1)\gamma t$ equal to unity in equation (3 (*a*) 8) we find, as expected, that $\epsilon V_{PQ} = 2L\sigma_0(s_0-1)/s_0$. Thus, when $s_0 > 1$, the potential between the plates falls steadily until a time T_c and thereafter remains constant.

In the particular case when $s_0 = 1$, the potential gradient at and between the points R′ and R is zero for all times and hence $(\mathrm{d}l/\mathrm{d}t)$ vanishes and l remains at its initial value. Setting $s_0 = 1$ in equation (3 (*a*) 6) we obtain the simple result

$$\epsilon V_{PQ}(t) = \rho_0 l_0^2/(1+\gamma t), \tag{3 (a) 9}$$

which means that the potential takes an infinite time to fall to zero in this case.

When $s_0 < 1$ the situation is a little more complicated because equation (3 (*a*) 8) only holds up to the time $T_e = [(L-l_0)/l_0\gamma(1-s_0)]$, when the charge clouds meet at the centre. Assuming that the charges recombine at the junction of the two clouds, we must, therefore, use equation (3 (*a*) 3) with l set equal to L. Hence,

$$\epsilon V_{PQ}(t) = \frac{2L\rho_0 l_0[s_0-(L/2l_0)]}{1+\gamma t} \quad (t > [(L-l_0)/l_0\gamma(1-s_0)] = T_e, \quad 1-(l_0/2L) < s_0 < 1). \tag{3 (a) 10}$$

It is noteworthy that equation (3(*a*) 10) indicates that $V_{PQ}(t)$ can become negative when $s_0 < 1$ if $s_0 < (L/2l_0)$. In addition, if $s_0 = (L/2l_0)$ we see that $V_{PQ}(t)$ vanishes for $t > T_e$. This means that at the instant when the charge clouds meet at the centre, the potential difference between the plates due to the space charge is exactly counterbalanced by that due to the charge on the plates.

The time dependence of $V_{PQ}(t)$ for various initial conditions having been determined, it is of interest to discuss, briefly, the motion of the charges in the charge clouds.

When the total charge on one plate exceeds the total space charge adjacent to it, that is when $s_0 > 1$ (see figure 1*b*), all parts of this space charge move towards the plate. However, the velocities of the charged particles increase, in a linear fashion, between R and Q because of the linear decrease in potential gradient between these points. The same is true in the special case when $s_0 = 1$ but in this case the potential gradient is zero at R (and at R′) but it still decreases linearly between R and Q. So it is that the width of the space charge cloud remains fixed whilst its magnitude falls with time.

The behaviour is more interesting when $s_0 < 1$ (figure 1*a*), because there is then a point D, between R and Q, within the space charge at which the potential gradient vanishes. In the region between R and D the potential gradient is positive and between D and Q it is negative; as a result charges between R and D move towards the centre, with speed increasing linearly from D to R, and those between D and Q move towards the plate with speed increasing linearly from D to Q.

The position of D, distant $w = \text{OD}$ from O, is determined by the equation

$$\rho(L-w) = \sigma.$$

Because of equations (3 (*a*) 2) and (3 (*a*) 3) it follows that

$$L-w = \sigma_0/\rho_0 = \text{constant},$$

and hence D is stationary.

In order to display the variation of $V_{\text{PQ}}(t)$ with time for various conditions we introduce a new variable $u = [1/(1+\gamma t)]$. With the dimensionless quantity $g(u) = [V_{\text{PQ}}(t)/V_{\text{PQ}}(0)]$, equation (3 (*a*) 8) may be written in the form

$$g(u) = \frac{u[s_0 - \{1+[(1-s_0)\,(1-u)/u]\} + \frac{1}{2}\kappa\{1+[(1-s_0)\,(1-u)/u]\}^2]}{s_0 - 1 + \frac{1}{2}\kappa}, \qquad (3\,(a)\,11)$$

where $\kappa = (l_0/L)$. This equation is valid under the following conditions (cf. equation (3 (*a*) 7)):

$$\left.\begin{array}{ll} 1 \geqslant u \geqslant (s_0-1)/s_0 & (s_0 > 1); \\ 1 \geqslant u \geqslant 0 & (s_0 = 1); \\ 1 \geqslant u \geqslant \kappa(1-s_0)/(1-\kappa s_0), & 1-\frac{1}{2}\kappa < s_0 < 1. \end{array}\right\} \qquad (3\,(a)\,12)$$

When $s_0 > 1$ and $u \leqslant (s_0-1)/s_0$ equation (3 (*a*) 11) must be replaced by

$$g(u) = (s_0-1)/[s_0 - 1 + \tfrac{1}{2}\kappa] = \text{constant}, \qquad (3\,(a)\,13)$$

and when $1-\frac{1}{2}\kappa < s_0 < 1$ and $u \leqslant \kappa(1-s_0)/(1-\kappa s_0)$, we must use the following relation derived from equation (3 (*a*) 10):

$$g(u) = \frac{u[s_0 - \frac{1}{2}\kappa]}{[s_0 - 1 + \frac{1}{2}\kappa]}, \quad u \leqslant \kappa(1-s_0)/(1-\kappa s_0). \qquad (3\,(a)\,14)$$

It should be noted that in this equation $g(u)$ can take on negative values.

Equations (3 (*a*) 11), (3 (*a*) 13) and (3 (*a*) 14) are shown in figure 2 for various values of s_0 and κ.

3 (*b*). *Short-circuit case*

We now investigate the behaviour of the charge system when, at time zero, the plates are brought to the same potential by means of an external connexion. In particular we shall be concerned with the current that flows in the external circuit and in the variations in the charge and field distributions.

The initial conditions in this case are the same as in the open-circuit case. Immediately on short circuit the potential difference between the plates, V_{PQ}, must fall to zero. The charge on a plate does not, however, immediately fall to zero. Before the short circuit the magnitude of the surface charge density on the plate is

$$\sigma(0-) = \epsilon V_{\text{PQ}}(0-)/2L + \rho_0 l_0[1-(l_0/2L)],$$

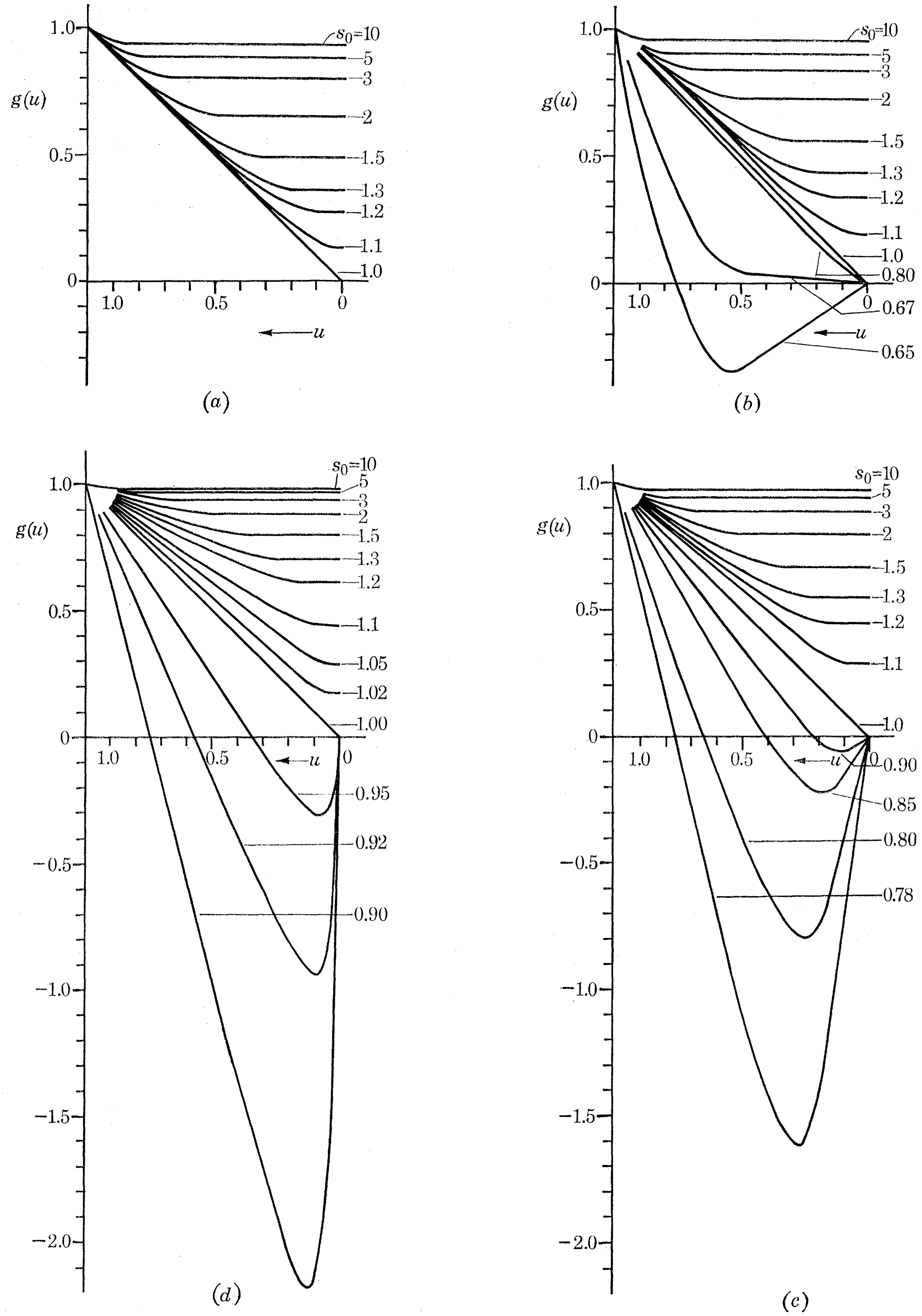

FIGURE 2. Open circuit. Relation between $g(u) = V_{PQ}(t)/V_{PQ}(0)$ and $u = (1+\gamma t)^{-1}$ for symmetrical charge distribution; (*a*) $\kappa = 1$; (*b*) $\kappa = 0.75$; (*c*) $\kappa = 0.50$; (*d*) $\kappa = 0.25$. (See equations (3(*a*), 11, 12, 13 and 14.)

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which is just equation (3 (*a*) 3) rearranged. Just after the short circuit, a part, $\epsilon V_{PQ}(0-)/2L$, of this charge density is removed, but a portion

$$\sigma(0+) = \rho_0 l_0[1-(l_0/2L)], \qquad (3\,(b)\,1)$$

remains. This initial sudden removal of the part of the surface charge density is effected by an external flow of current, that is, flow of positive charge from the left-hand plate to the right-hand plate. This is in the reverse direction to that of a normal charging current. However, as we shall see, the current flowing in the external circuit, thereafter, will be in the *same* direction as a normal charging current.

The field distribution just following the short circuit is shown in figure 1 (*c*). Near both plates the field is positive, that is, in the direction of the original applied field; it decreases linearly away from the plate and, after passing through zero at D′ and D it continues to decrease up to the points R′ and R, between which it is constant and negative. The corresponding potential distribution is shown in figure 1 (*c*).

With the particular initial charge distribution which we have chosen we know from our discussion in § 2 that, if both types of charge have the same mobility,

$$\rho(t) = \rho_0/(1+\gamma t), \qquad (3\,(b)\,2)$$

and that the shape of the space-charge distributions remains the same. In contrast to the open-circuit case the space charge clouds both tend to expand towards the centre, again without any spatial variation of ρ between P and R′ and between R and Q, no matter what value $\sigma(0-)/\rho_0 l_0$ may have, since there is a positive potential gradient at R′ and R. In fact,

$$dl/dt = -\mu E(L-l, t) = (\mu/2\epsilon)(\rho l^2/L)$$

and with equation (3 (*b*) 2) we find that

$$dl/dt = (\gamma/2L)\, l^2/(1+\gamma t),$$

which means that
$$l(t) = 2\kappa L/[2-\kappa \ln(1+\gamma t)] \quad (t \leqslant T_m), \qquad (3\,(b)\,3)$$

where the substitution, $\kappa = (l_0/L)$, has been made and where T_m, the time required for the charge clouds to meet at O, is determined by the equation

$$l(T_m) = L,$$

that is, when
$$\gamma T_m = \{\exp[2(1-\kappa)/\kappa]-1\}. \qquad (3\,(b)\,4)$$

From figure 1 (*c*) it is clear that the potential gradient vanishes at the points D′ and D and consequently the charges at these points will be stationary, but those between P and D′ and between D and Q move towards the plates, whereas the remaining charges move toward the centre. However, it is to be noted that in this case the points D′ and D are no longer stationary. Denoting the distance OD by ξ, we have, as the condition for the potential gradient, $-E(\xi, t)$, to vanish,

$$\rho(t)\,(L-\xi) = \sigma(t).$$

From equation (3 (*b*) 1) we deduce that

$$\xi(t) = L - l(t)\{1-[l(t)/2L]\} \quad (t \leqslant T_m), \qquad (3\,(b)\,5)$$

which, in view of equation (3 (*b*) 3), is a never increasing function of time.

The movement of D′ and D towards O means that some charges which initially moved towards the centre decelerate sufficiently so that the point D passes them and they then begin to move away from O and towards the plate. There is a point within the positive space charge to the left hand of which no charges ever reverse their motion. The distance, y, of this point from O is easily calculated from $\rho(T_m)$ because all the charge to the right of the point $x = \frac{1}{2}L = \xi(T_m)$ must always have been moving towards the centre, O. Therefore

$$\rho_0 y = \tfrac{1}{2}L\rho(T_m)$$

and so
$$y = L/[2(1+\gamma T_m)] = \tfrac{1}{2}L\exp[-2(1-\kappa)/\kappa].$$

The charge clouds meet when $l = L$, at $t = T_m$, and if recombination takes place immediately on contact of opposite charges, equation (3 (*b*) 3) ceases to be valid after the time T_m, given by equation (3 (*b*) 4). For values of $t > T_m$, the shape of the charge distribution remains fixed.

The velocity of the point D towards O, $-\mathrm{d}\xi/\mathrm{d}t$, satisfies the equation

$$-\xi' = -\mathrm{d}\xi/\mathrm{d}t = \{1-[l(t)/L]\}(\mathrm{d}l/\mathrm{d}t) \quad (t \leqslant T_m).$$

Therefore, $\xi'(T_m)$ vanishes and, from equation (3 (*b*) 5),

$$\xi = \tfrac{1}{2}L \quad (T_m \leqslant t \leqslant \infty).$$

In order to calculate the current induced in the external connexion due to motion of charges between the plates, we make use of the fact that a charge q moving towards the right-hand plate with speed v induces a flow of positive charge in the external connexion from the right plate to the left plate of amount $qv/2L$ (Shockley 1938). If, instead of a point charge, we consider a plane sheet of charge of surface density Σ then the current density leaving the right-hand plate is $(v\Sigma/2L)$.

We thus find that the contribution to the current density, $\mathrm{d}J^+$, leaving the right-hand plate, due to the motion of positive space charge between the planes x and $x+\mathrm{d}x$, satisfies the equation

$$\mathrm{d}J^+ = (\mu/2L)\,\rho(t)\,E(x,t)\,\mathrm{d}x. \qquad (3\,(b)\,6)$$

Because the electric field, $E(x,t)$, varies linearly within the space charge and is zero at $x = \pm\xi$, it is convenient to introduce the variable $\eta = x-\xi$. Then

$$\epsilon E(x,t) = \rho(t)\,\eta, \quad -l^2/2L < \eta < l[1-(l/2L)],$$

and integration of equation (3 (*b*) 6) yields

$$J^+ = \frac{\mu\rho^2}{2\epsilon L}\int_{-l^2/2L}^{l[1-(l/2L)]}\eta\,\mathrm{d}\eta = \frac{\mu\rho^2 l^2}{4\epsilon L}[1-(l/L)].$$

From the symmetry of the problem, J^-, the contribution from the motion of the negative charge cloud, must have the same value as J^+. The total current density, $J(t)$, for any time after the short circuit until $t = T_m$ is given by the relation

$$J(t) = \frac{2\kappa^2 L\rho_0\gamma}{\{(1+\gamma t)[2-\kappa\ln(1+\gamma t)]\}^2}\left\{1-\frac{2\kappa}{2-\kappa\ln(1+\gamma t)}\right\}, \quad \kappa = (l_0/L) \quad (0 < t \leqslant T_m). \qquad (3\,(b)\,7)$$

As a simplification we define a quantity τ, by the equation

$$(1+\gamma t) = \exp[2\tau(1-\kappa)/\kappa] \quad (0 < \tau \leqslant 1),$$

and hence
$$J(t) = \frac{(\kappa^2\rho_0\gamma L/2)[(1-\kappa)(1-\tau)]}{[1-\tau(1-\kappa)]^3}\exp[-4\tau(1-\kappa)/\kappa] \quad (\tau < 1).$$

As one would expect J vanishes at $t = T_m$ and for all later times when the flow of space charge towards the centre is exactly counterbalanced by the flow toward the plates. This means that although the external current vanishes after $t = T_m$, internal charge motion still persists. The space charge is removed by recombination at the plates and at the plane which divides the system; it will take an infinite time for the space charge to be completely removed according to equation (3 (*b*) 2).

For ease of computation it is helpful to use a dimensionless quantity, z, which has the same time dependence as J and which has unit value at $\tau = 0$. Thus,

$$z = J(t)/J(0) = \frac{(1-\tau)\exp[-4\tau(1-\kappa)/\kappa]}{[1-\tau(1-\kappa)]^3} \quad (0 < \kappa \leqslant 1;\ 0 < \tau \leqslant 1). \qquad (3\,(b)\,8)$$

The variation of z with τ is illustrated in figure 3 for various values of κ.

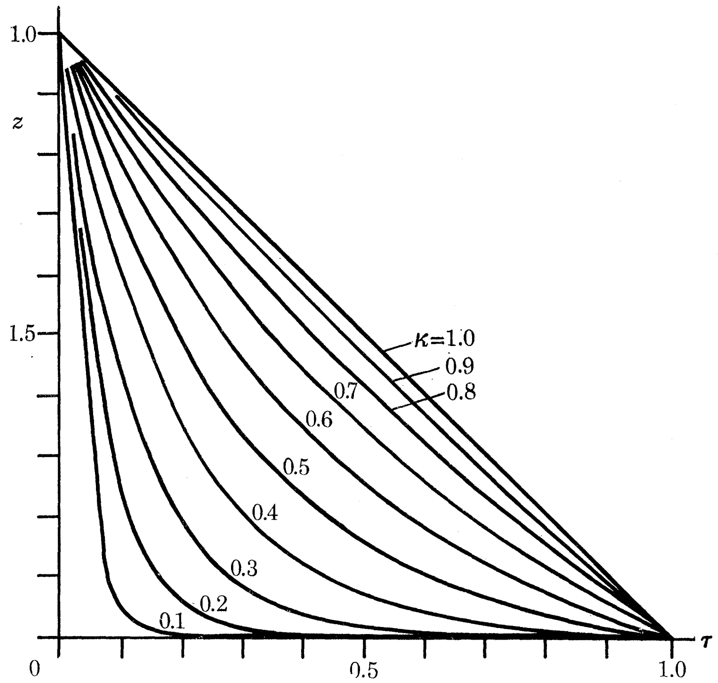

FIGURE 3. Short circuit. Relation between $z(\tau) = J(t)/J(0)$ and $\tau = [\kappa/2(1-\kappa)] \ln(1+\gamma t)$ for symmetrical charge distribution. (See equation (3 (*b*) 8).)

4. UNSYMMETRICAL CHARGE DISTRIBUTION

So far we have assumed that a symmetrical charge distribution exists and that the mobilities of the positive and negative charges are equal. In this section we shall show how it is possible to generalize, somewhat, the results of the previous section.

We shall no longer assume that the space charge distribution is symmetrical, but we shall still assume that the space charge density is spatially uniform in those regions where it exists. The notation which will now be used differs from that used in § 3 only to be extent that certain symbols will carry a superscript plus or minus sign as appropriate. The charge, field and potential distributions are shown in figure 4.

In order to make progress it has been found necessary to make the following, not unreasonable, assumptions.

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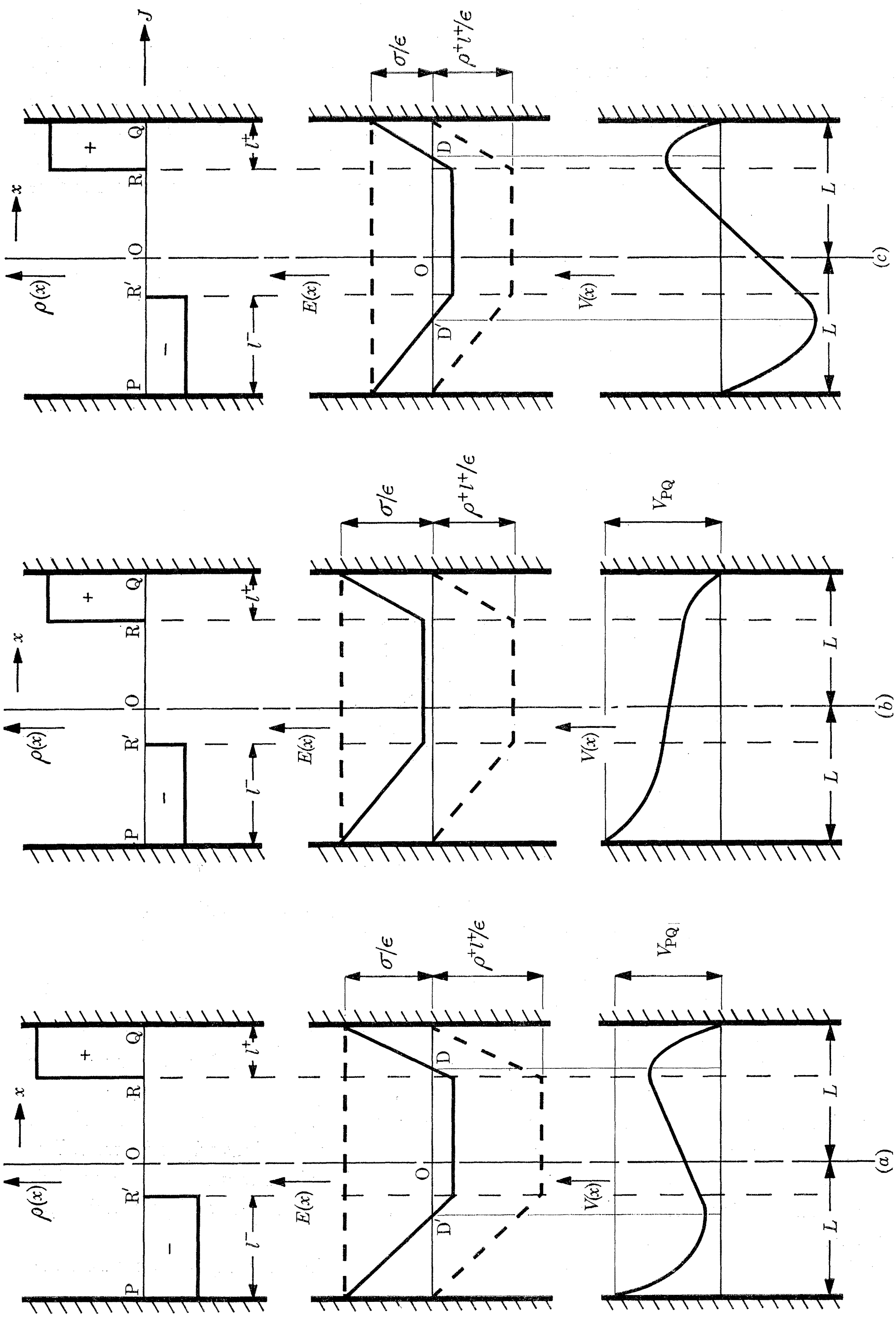

FIGURE 4. Distribution of charge, electric field and potential for unsymmetrical case; (*a*) open-circuit, $s_0 < 1$; (*b*) open-circuit; $s_0 > 1$; (*c*) short-circuit.

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(i) The total initial space charge between the plates is zero and therefore

$$\rho_0^+ l_0^+ = \rho_0^- l_0^- \tag{4.1}$$

and hence $\sigma_0^+ = -\sigma_0^- = \sigma_0$.

(ii) The initial space charge densities are related by the further condition that

$$\mu^+\rho_0^+ = \mu^-\rho_0^- = \gamma\epsilon. \tag{4.2}$$

The time dependences of ρ^+ and of ρ^- are both of the same form as indicated by equation (3 (*a*) 2) and so

$$\rho^\pm(t) = \rho_0^\pm/(1+\gamma t). \tag{4.3}$$

4 (*a*). *Open-circuit case*

The rate of decrease of surface charge density on a particular plate must be equal to the current density entering that plate; therefore,

$$\mathrm{d}\sigma^\pm/\mathrm{d}t = -\mu^\pm E(\pm L, t)\,\rho^\pm(t). \tag{4 (a) 1}$$

Because $\epsilon E(\pm L, t) = |\sigma^\pm(t)|$, equations (3 (*a*) 2), (3 (*a*) 3) and (4 (*a*) 1) lead to the formula

$$|\sigma^\pm(t)| = \sigma_0/(1+\gamma t) = \sigma(t). \tag{4 (a) 2}$$

By means of the same arguments which led to (3 (*a*) 6), we find that

$$\mathrm{d}l^\pm/\mathrm{d}t = -\mu^\pm E(\pm L \mp l^\pm, t)$$
$$= \mu^\pm[\rho^\pm(t)\,l^\pm(t) - \sigma(t)],$$

and, with $s_0 = \sigma_0/\rho_0^+ l_0^+ = \sigma_0/\rho_0^- l_0^-$,

we obtain

$$l^\pm(t) = l_0^\pm[1+(1-s_0)\,\gamma t]. \tag{4 (a) 3}$$

Notice that this result together with (4.3) implies that

$$\rho^+(t)\,l^+(t) = \rho^-(t)\,l^-(t). \tag{4 (a) 4}$$

For $s_0 > 1$, $l^\pm(t)$ is a decreasing function of time and equation (4 (*a*) 3) is only valid in the range $0 \leqslant t \leqslant [1/(s_0-1)\,\gamma] = T_c'$. The potential $V_{\mathrm{PQ}}(t)$ is constant for $t > T_c'$ because after this time all space charge has been removed from the dielectric. If $s_0 = 1$, then both l^+ and l^- are constant. When $s_0 < 1$, $l^\pm$ increase with time and equation (4 (*a*) 3) is then valid until a time T_e' when the charge clouds meet, that is when $l^+(T_e') + l^-(T_e') = 2L$. As a consequence

$$T_e' = [2L-(l_0^+ + l_0^-)]/[(l_0^+ + l_0^-)\,\gamma(1-s_0)],$$

and

$$l^\pm(T_e') = 2l_0^\pm L/(l_0^+ + l_0^-).$$

Equation (4 (*a*) 3) is, therefore, valid under the following conditions:

$$\left.\begin{array}{l} 0 \leqslant t \leqslant [1/\gamma(s_0-1)] = T_c', \quad s_0 > 1; \\ 0 \leqslant t \leqslant \infty, \quad s_0 = 1; \\ 0 \leqslant t \leqslant T_e', \quad [1-(l_0^+ + l_0^-)/4L] < s_0 < 1. \end{array}\right\} \tag{4 (a) 5}$$

The potential difference between the plates, $V_{\mathrm{PQ}}(t)$, satisfies the relation

$$\epsilon V_{\mathrm{PQ}}(t) = 2L\sigma - [2L-(l^+ + l^-)/2]\,\rho^+ l^+, \tag{4 (a) 6}$$

which may easily be obtained by calculating the total area beneath the plot of E against x in figure 4 (*a*) or 4 (*b*). The lower bound for s_0 is determined from equation (4 (*a*) 6) by the condition that $V_{PQ}(0) > 0$.

Substituting for $\rho^{\pm}(t)$, $l^{\pm}(t)$ and $\sigma(t)$ in equation (4 (*a*) 6) from equations (4.3), (4 (*a*) 3) and (4 (*a*) 2), we obtain

$$\epsilon V_{PQ}(t) = \frac{2L\rho_0^+ l_0^+}{1+\gamma t}\left\{s_0 - [1+(1-s_0)\,\gamma t] + \frac{(l_0^+ + l_0^-)}{4L}[1+(1-s_0)\,\gamma t]^2\right\}. \qquad (4\,(a)\,7)$$

This equation is valid for the range of values of t and s_0 quoted in (4 (*a*) 5).

If we assume that recombination takes place at the junction between the two clouds of space charge, there is no overflow of charge of one sign into the region occupied by charge of the opposite sign because the current densities J_j^+ and J_j^-, on either side of the junction, are equal. This may be proved as follows:

$$J_j^+ = \rho^+\mu^+(\rho^+l^+ + \rho^-l^-)/2\epsilon,$$

and

$$J_j^- = \rho^-\mu^-(\rho^+l^+ + \rho^-l^-)/2\epsilon,$$

but, from equations (4.2) and (4.3), $\rho^+\mu^+ = \rho^-\mu^-$ and hence $J_j^+ = J_j^-$. A corollary to this result is that the position of the junction of the space charges remains fixed.

The variation of $V_{PQ}(t)$ described by equation (4 (*a*) 7) is essentially the same as for the symmetrical case. We notice that for $s_0 = 1$,

$$\epsilon V_{PQ}(t) = \rho_0^+ l_0^+ (l_0^+ + l_0^-)/[2(1+\gamma t)], \qquad (4\,(a)\,8)$$

for all values of t, and that when $s_0 < 1$,

$$\epsilon V_{PQ}(t) = \frac{2L\rho_0^+ l_0^+}{1+\gamma t}[s_0 - L/(l_0^+ + l_0^-)] \qquad (4\,(a)\,9)$$

for $t > T_e'$. In this last case the condition $1-(l_0^+ + l_0^-)/4L < s_0 < 1$ must be satisfied to ensure that $V_{PQ}(0)$ is positive. Just as in the symmetrical case $V_{PQ}(t)$ can take on negative values.

The comments at the end of § 3 (*a*) concerning the motion of the space charges apply with appropriate, but obvious, changes.

4 (*b*). *Short-circuit case*

The considerations in § 3 (*b*) will now be repeated, in outline, for the unsymmetrical charge distribution. The charge density on one of the plates, $\sigma(0-)$, just preceding short circuit, satisfies the equation (see figure 4 (*a*) or 4 (*b*))

$$\sigma(0-) = [\epsilon V_{PQ}(0-)/2L] + [1-(l_0^+ + l_0^-)/4L]\,\rho_0^+ l_0^+.$$

On short circuit the charge density falls to a value $\sigma(0+)$ given by

$$\sigma(0+) = \rho_0^+ l_0^+[1-(l_0^+ + l_0^-)/4L].$$

It then follows that

$$dl^{\pm}/dt = (\mu^{\pm}/4L\epsilon)\,[\rho^+(l^+)^2 + \rho^-(l^-)^2], \qquad (4\,(b)\,1)$$

and hence

$$l^+(t) - l_0^+ = (\mu^+/\mu^-)\,[l^-(t) - l_0^-].$$

In view of equations (4.1) and (4.2) this means that

$$\mu^- l^+(t) = \mu^+ l^-(t), \qquad (4\,(b)\,2)$$

and so equation (4 (*b*) 1) may be integrated to obtain the result

$$l^{\pm}(t) = l_0^{\pm}/\{1-[(\mu^+ + \mu^-)/4\mu^+ L]\, l_0^{\pm} \ln(1+\gamma t)\}. \qquad (4\,(b)\,3)$$

It is clear that both $l^+(t)$ and $l^-(t)$ are increasing functions of time as was $l(t)$ in the symmetrical case. Equation (4 (*b*) 3) ceases to be valid when the space charges meet, that is when $l^+ + l^- = 2L$. From equations (4 (*b*) 2) and (4 (*b*) 3) the time of meeting, T'_{m}, is obtained by solving the equation

$$\frac{2L\mu^+}{(\mu^+ + \mu^-)} = \frac{l_0^+}{\{1-[(\mu^+ + \mu^-)\, l_0^+/4\mu^+ L] \ln(1+\gamma t)\}}.$$

Therefore,

$$(1+\gamma T'_{\mathrm{m}}) = \exp[-2+4L\mu^+/(\mu^+ + \mu^-)\, l_0^+]. \qquad (4\,(b)\,4)$$

As before, if we introduce a dimensionless quantity τ' defined by the relation

$$(1+\gamma t) = \exp[2\tau'(1-\kappa')/\kappa'],$$

where

$$\kappa' = (\mu^+ + \mu^-)\, l_0^+/2L\mu^+,$$

then

$$l^{\pm}(t) = 2\kappa' l_0^{\pm}/[2-\kappa' \ln(1+\gamma t)] \quad (t \leqslant T'_{\mathrm{m}}). \qquad (4\,(b)\,5)$$

Furthermore, one can prove that the current density in the external circuit, $J(t)$, is given by the formula

$$J(t) = \frac{[4\kappa'^2 \rho_0^+ \gamma\mu^+/(\mu^+ + \mu^-)]}{\{(1+\gamma t)\,[2-\kappa' \ln(1+\gamma t)]^2\}} \left\{1-\frac{2\kappa'}{[2-\kappa' \ln(1+\gamma t)]}\right\}, \qquad (4\,(b)\,6)$$

which is of the same form as equation (3 (*b*) 7). It will be seen from equation (4 (*b*) 6) that $J(t)$ vanishes for $t = T'_{\mathrm{m}}$ and it may be shown that $J(t)$ is zero for all later times.

The essential features of the motion of the space charge in the symmetrical case, discussed in § 3 (*b*), are maintained in this case also. Thus the curves illustrated in figures 2 and 3 apply to a particular asymmetrical charge distribution as well as to the symmetrical distribution discussed in § 3.

5. Discussion

The foregoing analysis is based upon a very simple physical model. At best it is only likely to be an approximate representation of the true physical situation. Such factors as diffusion, dielectric relaxation, intrinsic conduction, bulk movement of the fluid, and effects peculiar to the electrode/fluid interface have been ignored. Nevertheless, the simple model enables quite complicated transient voltage and current behaviour to be predicted, including behaviour resembling that observed experimentally. Quite sophisticated models are sometimes postulated to account for such observations, but the possibility of accounting for them by using a simple model should be considered.

For example it may be predicted, using our simple model, that after short circuit of a dielectric specimen the current flow will be in the same direction as current flow before the short circuit is effected, apart from a sudden pulse of current in the opposite direction immediately on short circuit. This is in agreement with the observations on irradiated *n*-hexane (Yahagi *et al.* 1966). It is plausible to suggest that under irradiation the space charge, like the steady-state current, is very considerably greater than in the non-irradiated case. This being so, it may well be that our simple model is a reasonably good representation of the actual situation obtaining. On the other hand, this model is not likely to be an acceptable representation of the situation obtaining in the non-irradiated case, because the direction of current flow predicted after short circuit is opposite to that actually observed. Some other mechanism must be postulated, which may also be operative in the irradiated case when it is swamped by the greater space charge effect.

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